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Gravitation and Electrodynamics Over SO(3,3)

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In a series of papers, an approach to field theory is developed in which matter appears by interpreting source-free (homogeneous) fields over a 6-dimensional space of signature (3,3), as interacting (inhomogeneous) fields in space-time. The extra dimensions are given a physical meaning as "coordinatized matter." The inhomogeneous energy-momentum relations for the interacting fields in space-time are automatically generated by the simple homogeneous relations in 6-d. We then develop a Weyl geometry over SO(3,3) as base, under which gravity and electromagnetism are essentially unified via an irreducible 6-calibration invariant Lagrange density and corresponding variational principle. The Einstein–Maxwell equations are shown to represent a low-order approximation, and the cosmological constant must vanish in order that this limit exist.

KEY WORDS:

1. OVERVIEW

The old goal of understanding the long-range forces on a common basis remains a compelling one. The classical attacks on this problem fell into four classes:

1. Projective theories (Kaluza, Pauli, Klein)
2. Theories with asymmetric metric (Einstein–Mayer)
3. Theories with asymmetric connection (Eddington)
4. Alternative geometries (Weyl)

All these attempts failed. In one way or another, each is reducible and thus any unification achieved is purely formal. The Kaluza theory requires an adhoc hypothesis about the metric in 5-d, and the unification is nondynamical. As Pauli showed, any generally covariant theory may be cast in Kaluza's form. The Einstein–Mayer theory is based on an asymmetric metric, and as with the theories based on asymmetric connection, is essentially algebraically reducible without additional, purely formal hypotheses.

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Weyl's theory, however, is based upon the simplest generalization of Riemannian geometry, in which both length and direction are nontransferable. It fails in its original form due to the nonexistence of a simple, irreducible calibration invariant Lagrange density in 4-d. One might say that the theory is *dynamically* reducible. Moreover, the possible scalar densities lead to 4th order equations for the metric, which, even supposing physical solutions could be found, would be differentially reducible. Nevertheless the basic geometrical conception is sound, and given a suitable Lagrangian and variational principle, leads almost uniquely to an essential unification of gravitation and electrodynamics with the required source fields and conservation laws.

What characteristics would a proper unification of gravity and electromagnetism, that is, the metric and vector potential fields, possess, in order that they both appear as essential parts of a larger whole, while preserving their unique character as physical fields?

- It must lead to an essential coupling in terms of the differential equations that the A and g field obey.
- It must reduce to ordinary gravity alone when A is a gradient, and so the physical electromagnetic field vanishes.
- It must reduce to ordinary electrodynamics in the extreme weak field limit—that is, the inhomogeneous Maxwell theory on essentially flat space.
- The equations must be no higher than second-order in the g-field.
- The equations must be no higher than second-order in the physical electromagnetic field derived from the A-field, thus third-order in A.
- The conservation of charge and energy-momentum should be jointly derivable from a variational principle by Noether's theorem.

In what follows, a theory satisfying all these requirements is developed.

2. VACUUM ELECTRODYNAMICS IN SIX DIMENSIONS

Notation. Roman letters go 1–6, other than i, j, k which go 1–3. Greek letters are generic, or go 1–4 in the context of space-time. The spatial entries in the metric carry negative signature. We often refer to $x5$ and $x6$ as u and v, respectively.

We start by briefly considering the analog of the homogeneous Maxwell equations on SO(3,3). The field equations are

$$\partial_n F^{mn} = 0, \quad \partial_p F_{mn} + \partial_m F_{np} + \partial_n F_{pm} = 0 \tag{1}$$

$$F^{mn} = \begin{bmatrix} F^{\mu\nu} & -S^{\mu} & -T^{\mu} \\ S^{\mu} & 0 & -\eta \\ T^{\mu} & \eta & 0 \end{bmatrix} \tag{2}$$

where for clarity we have written out the field strength tensor in terms of 4-d covariants. We see that there is the usual Maxwell field tensor, a 4-vector and a 4-pseudovector, and a pseudoscalar (see below), if we consider only those *proper* transformations of SO(3,3) that leave the 1–4 and 5–6 subspaces invariant. In terms of these components the field equations are

$$\begin{aligned} F^{\mu\nu}_{,\nu} &= \partial_u S^\mu + \partial_v T^\mu = J^\mu \\ \partial_\mu S^\mu &= -\partial_v \eta \\ \partial_\mu T^\mu &= \partial_u \eta \end{aligned} \tag{3}$$

which are analogous to the “first set” in Maxwell’s equations, and

$$\begin{aligned} \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} &= 0 \\ \partial_u F_{\mu\nu} &= \partial_\mu S_\nu - \partial_\nu S_\mu \\ \partial_v F_{\mu\nu} &= \partial_\mu T_\nu - \partial_\nu T_\mu \\ \partial_\mu \eta &= \partial_u T_\mu - \partial_v S_\mu = K_\mu \end{aligned} \tag{4}$$

which are analogous to the “second set.” The first set has 6 equations, the second has 20. We have introduced the convenient shorthands J and K for the indicated combinations of derivatives of the S and T components of the field.

Because we restrict ourselves to the *proper* transformations of SO(3,3), we see that a t-reversal must be accompanied by either a u-reversal or a v-reversal, *but not both*, in order to have a proper transformation of the entire 6-d space. This implies that, from the perspective of the Lorentz group, if u is a scalar, then v is a pseudoscalar. We adopt this convention now, without loss of generality. Thus J becomes the sum of two space-time vectors, K the sum of two space-time pseudovectors, and finally η is a pseudoscalar itself—and this latter is independent of the handedness convention we employ.

We see that the equations above have the form of the *inhomogeneous* Maxwell equations, with additional equations for the field η that establish a geometrical relation between the component vectors S and T as they appear in the current. Note that neither S nor T is conserved in the 4-d sense, but the 4-vector J made from the *sum* of derivatives of S and T with respect to u and v, respectively is indeed conserved in the 4-d sense. Thus we have a surprising alternate representation of the inhomogeneous Maxwell theory by interpreting 6-d covariants from the perspective of space-time. The “dual current” K is a pseudovector and has vanishing 4-curl, which gives nearly unambiguous information as to its interpretation—the Pauli–Lubanski spin vector. The extra fields allow something unexpected on the classical level, the introduction of spin. We go further and interpret the summands in the 4-current as representations of *matter* and *antimatter*. This makes sense, in that we may represent a parity operation in 4-d as a continuous operation in the proper 6-d group—thus a half-rotation of the "time-space" about the u-axis reverses both t and v. Further, we know from experience that neither matter nor antimatter *alone* is conserved, rather, only the *sum* of the two regarded as *proper* 4-vectors pointing forward in time. Thus the thorny issue of interpretation of negative energy that arises in the Dirac theory, is alleviated from the very beginning in this formalism. Matter and antimatter are fully and equally represented as *logically independent entities*, that are related by field equations.

The equations at first sight are highly overdetermined—we have 26 equations for 15 field variables. However, just as in the 4-d case, this is not so, and we may interpret some of the equations as initial conditions on the fields. Thus, we select one of the time-like variables, say $u = x5$, and ask how many equations in both sets do not involve derivatives with respect to it. In the first set the answer is obviously 1, while in the second set, no index may be 5 because of total antisymmetry, giving C(6–1,3) = 10 equations. This gives 11 equations in total that do not involve u—by manipulating the remaining 15 equations, we may express these 11 in the form of initial-u conditions. This leaves finally 15 equations for 15 unknowns, so the system is well-determined. (This analysis actually applies in any number of dimensions.)

Because of total antisymmetry, the second set implies that the field strength tensor may be derived from a potential:

$$F_{mn} = \partial_m A_n - \partial_n A_m \tag{5}$$

in terms of which the various components are expressed

$$\begin{aligned} &B = \nabla \times A \quad E = -\nabla_\varphi - \partial_t A \\ &S = -\nabla\chi - \partial_u A \quad \sigma = \partial_t \chi - \partial_u \varphi \\ &T = -\nabla_\psi - \partial_v A \quad \tau = \partial_t \psi - \partial_v \varphi \\ &\eta = \partial_u \psi - \partial_v \chi \end{aligned} \tag{6}$$

In falt space with Cartesian coordinates, the components of the potential all satisfy the ultrahyperbolic wave equation

$$\left(\partial_u^2 + \partial_v^2 + \partial_t^2 - \nabla^2\right) A = 0 \tag{7}$$

In terms of this potential, the currents have the expressions

$$\begin{aligned} J^\mu &= -\left(\partial_u^2 + \partial_v^2\right) A^\mu + \partial^\mu (\partial_u \chi + \partial_v \psi) \\ &= \left(\partial_t^2 - \nabla^2\right) A^\mu - \partial^\mu (\nabla \cdot A + \partial_t \phi) \\ K^\mu &= \partial^\mu (\partial_u \psi - \partial_v \chi) \end{aligned} \tag{8}$$

where we assumed the 6-covariant condition on the potentials

$$\partial_m A^m = 0 \tag{9}$$

Notice that these expressions are just those of ordinary electrodynamics, before imposition of the 4-covariant Lorentz gauge. In the present formalism, the 4-covariant Lorentz gauge corresponds to a special gauge that is 4- but not 6-covariant:

$$\partial_\mu A^\mu = 0 \quad \partial_u \chi + \partial_v \psi = 0 \tag{10}$$

It may be verified that the following tensor is symmetric and conserved (but not as in 4-d traceless):

$$T^{mn} = F^{ma} F_a{}^n + \frac{1}{4} g^{mn} F^{ab} F_{ab} \quad \partial_n T^{mn} = 0 \tag{11}$$

This is the 6-d statement of energy-momentum conservation. We split this 6-d statement into $4 + 2$ form as follows:

$$\begin{aligned} T^{\mu\nu} &= F^{\mu\alpha} F_\alpha{}^\nu + F^{\mu 5} F_5{}^\nu + F^{\mu 6} F_6{}^\nu + \frac{1}{4} g^{\mu\nu} F^{ab} F_{ab} \\ &= T^{\mu\nu}_{\text{Maxwell}} - S^\mu S^\nu - T^\mu T^\nu + \frac{1}{2} g^{\mu\nu} (S^\alpha S_\alpha + T^\alpha T_\alpha + \eta^2) \end{aligned} \tag{12}$$

$$\begin{aligned} T^{\mu 5} &= F^{\mu\alpha} F_\alpha{}^5 + F^{\mu 6} F_6{}^5 \\ &= -F^{\mu\alpha} S_\alpha - \eta T^\mu \end{aligned} \tag{13}$$

$$\begin{aligned} T^{\mu 6} &= F^{\mu\alpha} F_\alpha{}^6 + F^{\mu 5} F_5{}^6 \\ &= -F^{\mu\alpha} T_\alpha - \eta S^\mu \end{aligned} \tag{14}$$

$$\begin{aligned} T^{55} &= \frac{1}{2}(-\eta^2 + B^2 - E^2 - S^\mu S_\mu + T^\mu T_\mu) \\ T^{66} &= \frac{1}{2}(-\eta^2 + B^2 - E^2 - S^\mu S_\mu + T^\mu T_\mu) \end{aligned} \tag{15}$$

$$T^{56} = -S^\mu T_\mu \tag{16}$$

$$\mathit{trace}\,(T) = B^2 - E^2 + \eta^2 + S^\mu S_\mu + T^\mu T_\mu = \frac{1}{2} F^{mn} F_{mn} \tag{17}$$

After some manipulation using the field equations, we arrive at

$$\partial_\nu T^{\mu\nu}_{\text{Maxwell}} = F^{\mu\alpha}(\partial_u S_\alpha + \partial_v T_\alpha) = F^{\mu\alpha} J_\alpha \tag{18}$$

which are the energy relations in space-time for an *inhomogeneous* field. Thus, a solution to the 6-d vacuum equations is seen from the perspective of space-time as both the Maxwell field equations with sources, as well as the action of the field on that source—in other words, the complete Maxwell–Lorentz dynamical system. One can, say, take a 6-plane wave solution and interpret from space-time–the current will be a continuous “charge wave” filling all space, moving with speed *less than c*—this is a consequence of the time-like choice of the extra dimensions.

This charge wave is self-sustaining and nondispersive by virtue of the dynamical relations. By a superposition of charge waves, one may build up any charge distribution with Fourier analysis and get the required dynamics automatically. This may have practical value as a new method for solving problems in applied electrodynamics.

There are two more energy relations to consider, and one might at first guess that they produce energy relations involving the dual current K—but in fact they are simply identities and contribute nothing to space-time dynamics. So the role of the dual current seems somewhat mysterious. Although out of the main line of development here, a few words are in order about its role—more detail is provided in companion papers where we consider the Dirac theory extended to SO(3,3).

First, note that when the pseudoscalar field is constant, the dual current vanishes and the currents of matter $\partial_u S^\mu$ and antimatter $\partial_v T^\mu$ are separately conserved—there is no creation or annihilation possible and the distinction between the two is purely conventional. This dependence can be reversed—assume

$$\partial_\mu(\partial_u S^\mu) = -\partial_u \partial_v \eta = 0 \tag{19}$$

Thus η is independent of u and v. But this implies

$$\partial_\mu K^\mu = \left(\partial_u^2 + \partial_v^2\right)\eta = 0 \tag{20}$$

Now the 4-vector K^μ has both vanishing curl and divergence, and so by the extension of Helmholtz' theorem to space-time, is constant and in fact may be assumed to be zero. But this in turn implies that η is independent of x and t as well as u and v. Thus the changes in η from place to place measure the local rate of creation/annihilation. The dual current thus establishes a constraint on the relation of matter to antimatter. In order to deal with the charge current in total, we require both J and K.

With this reformulation of electrodynamics, let us proceed to the main topic, the creation of a Weyl field theory in 6-d with the purpose of unifying the g and A fields under the principle of "pure infinitiesimal geometry."

3. RÉSUMÉ OF THE WEYL GEOMETRY

Shortly after the creation of general relativity, Hermann Weyl showed that a simple generalization of Riemannian geometry, in which both length and direction are nontransferable, required the introduction of a vector field and differential identities, which gave jointly the conservation of the energy tensor and the charge current as a consequence of the geometry.

The basic physical assumption is that the light cone has physical primacy for phenomena—one gives up the idea that the space-time interval itself is invariant unless that interval is zero. This amounts to the idea that the local standard of measurement, the "calibration," is itself subject to the action of physical fields.

Riemannian geometry, of course, preserves lengths because the covriant derivative of the metric tensor is zero. Thus, to accommodate the idea of a local calibration, we must find a geometry that reduces to Riemannian geometry when the calibration field may be globally determined.

The simplest mathematical assumption is that the change of calibration in an infinitesimal region is proportional to the locally determined length, the same proportion holding regardless of the length. We assume that the space is symmmetrically connected. Then we have the expression for the change of the length of a vector under an infinitesimal displacement

$$\delta(V^\mu V_\mu) = -\delta\lambda(x^m)(V^\mu V_\mu) \tag{21}$$

$$\begin{aligned} \delta V^\mu &= -\Gamma^\mu_{\alpha\beta}\delta x^\alpha V^\beta \\ \Gamma^\mu_{\alpha\beta} &= \Gamma^\mu_{\beta\alpha} \end{aligned} \tag{22}$$

Assuming a change in calibration is linear and homogeneous in the displament,

$$\begin{aligned} \delta\lambda &= A_\mu \delta x^\mu \\ 2g_{\mu\nu}V^\mu\delta V^\nu + \delta g_{\mu\nu}V^\mu V^\nu &= -(\Gamma_{\mu\alpha\beta}\delta x^\alpha + \Gamma_{\beta\alpha\mu}\delta x^\alpha - \delta g_{\mu\beta})V^\mu V^\beta \\ &= -A_\alpha\delta x^\alpha g_{\mu\beta}V^\mu V^\beta \end{aligned} \tag{23}$$

Because V is arbitrary, we get a relation between the calibration field A and the metric:

$$(\Gamma_{\mu\alpha\beta} + \Gamma_{\beta\alpha\mu})\delta x^\alpha = \delta g_{\mu\beta} + A_\alpha g_{\mu\beta}\delta x^\alpha \tag{24}$$

and so

$$\Gamma_{\mu\alpha\beta} + \Gamma_{\beta\alpha\mu} = \partial_\alpha g_{\mu\beta} + A_\alpha g_{\mu\beta} \tag{25}$$

Solving for the connection yields

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta} + A_\alpha g_{\mu\beta} + A_\beta g_{\mu\alpha} - A_\mu g_{\alpha\beta}) \tag{26}$$

It follows that the covariant derivative of the metric is not zero, rather

$$g_{\mu\nu;\alpha} = -g_{\mu\nu}A_\alpha \quad g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}A_\alpha \tag{27}$$

Let us write a calibration change as

$$g_{\mu\nu} \to \lambda g_{\mu\nu} \quad g^{\mu\nu} \to \frac{1}{\lambda}g^{\mu\nu} \tag{28}$$

with the additional hypothesis

$$A_\alpha \to A_\alpha - \partial_\mu \log\lambda = A_\alpha - \frac{1}{\lambda}\partial_\mu\lambda \tag{29}$$

The connection is invariant under this joint change. We arrive at a generalized tensor analysis in which objects have covariance under both coordinate and calibration changes considered as logically independent operations. Candidate tensors will get a factor of λ on change of calibration:

$$T^{\alpha\beta\cdots}_{\mu\nu\cdots} \to \lambda^{W} T^{\alpha\beta\cdots}_{\mu\nu\cdots} \tag{30}$$

The factor W is called the *weight* of the object. Absolute invariants are those having $W = 0$. The covariant metric tensor has weight $+1$, the contravariant -1. The field strength tensor formed from the curl of A is absolutely invariant. A covariant of weight $+1$ is often simply referred to as a *tensor density* proper. Observables are typically tensor densities.

One may now form the curvature tensor from the connection in the usual way, by commutation of covariant derivatives, which now becomes an expression in both g and A and their derivatives. We adopt the convention

$$R^{\rho}_{\mu\alpha\beta} = \partial_\alpha \Gamma^{\rho}_{\mu\beta} - \partial_\beta \Gamma^{\rho}_{\mu\alpha} + \Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\mu\beta} - \Gamma^{\rho}_{\beta\sigma}\Gamma^{\sigma}_{\mu\alpha} \tag{31}$$

The curvature tensor is calibration invariant and has the following symmetry properties common to all symmetric connections:

$$\begin{aligned} R^{\rho}_{\mu\alpha\beta} + R^{\rho}_{\mu\beta\alpha} &= 0 \\ R^{\rho}_{\mu\alpha\beta} + R^{\rho}_{\alpha\beta\mu} + R^{\rho}_{\beta\mu\alpha} &= 0 \\ R^{\rho}_{\mu\alpha\beta;\gamma} + R^{\rho}_{\mu\beta\gamma;\alpha} + R^{\rho}_{\mu\gamma\alpha;\beta} &= 0 \end{aligned} \tag{32}$$

The covariant curvature tensor is however not antisymmetric in the first indices, nor is it symmetric under exchange of its first and last index pair. In fact

$$R_{\alpha\beta\mu\nu} + R_{\beta\alpha\mu\nu} = g_{\alpha\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu) = g_{\alpha\beta} F_{\mu\nu} \tag{33}$$

Weyl suggestively refers to this a resolution into “direction curvature” and “length curvature.”

The contracted curvature tensor is also not symmetric—a simple calculation shows

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha} \quad R_{\mu\nu} - R_{\nu\mu} = -\frac{1}{2} N F_{\mu\nu} \tag{34}$$

where N is the dimension of the base space. The contracted tensor, like the full tensor, is absolutely invariant. Note that even the symmetric part of the contracted tensor involves the A field—a further contraction yields

$$R = R^{\alpha}_{\alpha} = R^{*} + (N-1)\frac{1}{\sqrt{}}\partial_\mu(\sqrt{}A^{\mu}) + \frac{(N-1)(N-2)}{4} A^{\mu} A_{\mu} \tag{35}$$

where this starred scalar is that part of R involving no A terms. The curvature scalar is weight -1. Here the bare square root sign is shorthand for “square root of the determinant of g,” a very handy notation invented by Dirac.

The absent symmetries of the curvature tensor require careful treatment of the contracted Bianchi identities. We start with the full identities and pull the metric underneath the covariant differentiation, remembering that the covariant derivative of the metric is not zero:

$$R^{\mu\nu}_{\alpha\beta;\gamma} + R^{\mu\nu}_{\beta\gamma;\alpha} + R^{\mu\nu}_{\gamma\alpha;\beta} = g^{\nu\rho}_{;\gamma} R^{\mu}_{\rho\alpha\beta} + g^{\nu\rho}_{;\alpha} R^{\mu}_{\rho\beta\gamma} + g^{\nu\rho}_{;\beta} R^{\mu}_{\rho\gamma\alpha}$$
$$= A_\gamma R^{\mu\nu}_{\alpha\beta} + A_\beta R^{\mu\nu}_{\gamma\alpha} + A_\alpha R^{\mu\nu}_{\beta\gamma} \tag{36}$$

Put $\beta = \mu$;

$$R^{\nu}_{\alpha;\gamma} - R^{\nu}_{\gamma;\alpha} + R^{\mu\nu}_{\gamma\alpha;\mu} = A_\gamma R^{\nu}_{\alpha} - A_\alpha R^{\nu}_{\gamma} + A_\mu R^{\mu\nu}_{\gamma\alpha} \tag{37}$$

and now $\gamma = \nu$ and use the symmetry properties of the Ricci tensor in this space:

$$2R^{\nu}_{\alpha;\nu} - R_{;\alpha} + F^{\nu}_{\alpha;\nu} = 2A_\nu R^{\nu}_{\alpha} - A_\alpha R + A_\nu F^{\nu}_{\alpha} \tag{38}$$

and now raise index α by again pulling the metric under the covariant derivative:

$$2R^{\nu\alpha}_{;\nu} - (g^{\nu\alpha} R)_\nu + F^{\nu\alpha}_{;\nu} = 4A_\nu R^{\nu\alpha} - 2A_\nu g^{\nu\alpha} R + 2A_\nu F^{\nu\alpha} \tag{39}$$

and so finally we arrive at

$$(\nabla_\alpha - 2A_\alpha)\left(R^{\nu\alpha} - \frac{1}{2} g^{\nu\alpha} R + \frac{1}{2} F^{\nu\alpha}\right) = 0 \tag{40}$$

An equivalent form is

$$\nabla^\alpha \left(R_{\nu\alpha} - \frac{1}{2} g_{\nu\alpha} R + \frac{1}{2} F_{\nu\alpha}\right) = 0 \tag{41}$$

The explanation of the A term on the left of the first form is found in the concept of *conformal covariant derivative* of a tensor of weight N;

$$D_\alpha T_{(N)} = (\nabla_\alpha + N A_\alpha) T_{(N)} \tag{42}$$

We now have a formalism we can apply to a joint field theory for the A and g. To proceed to physics, we must make assumptions about the the possible Lagrange functions that might appear in a variational principle. We assume the integrand will be an absolute invariant—the field equations are then guaranteed to be calibration as well as coordinate invariant. Now, the dimension of the base space comes into play, because to make an invariant integrand, we must take an absolute scalar and append a factor of sqrt(det(g)) to make an invariant volume element in the integration. In space-time, the determinant involves 4 factors of covariant metric components, and on taking the square root, we see that the weight of the prefactor is 2. Thus, the possible scalars must be of weight -2. The only simple scalars of weight -2 in space-time are

$$R^2,\ R^{\mu\nu} R_{\mu\nu},\ R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta},\ F^{\mu\nu} F_{\mu\nu} \tag{43}$$

We see now that we are in trouble in 4-d. The first three will lead to 4th order differential equations in the metric. Even if these may be differentially reduced to match up with gravitation as we know it, we cannot be satisfied with such a process, and in all likelihood there is a large excess of unphysical solutions at hand. The last expression does not lead to equations for the metric at all. Weyl himself tried the expression

$$R^2 + \kappa F^{\mu\nu} F_{\mu\nu} \tag{44}$$

Now however, we have an even more serious objection—the arbitrary multiplicative factor wrecks the unification at the start. After a promising beginning, the theory fails for the simplest of reasons—space-time is simply the wrong dimension. In hindsight, we would have been in trouble in any case, because nowhere does the electric current appear—thus it would never be possible for this theory to reduce to ordinary electrodynamics in the absence of gravity. The best that could be achieved would be a world with no charges and free light waves, having nothing to shine on.

Shortly after it was published, Einstein showed that Weyl's theory made predictions that were at odds with experience. Namely, it would not be possible for sharp spectral lines to exist, because a naive application of the idea to a moving electron shows that its mass is dependent on its space-time history. This physical failure corresponds exactly to the mathematical failure of the missing variational integrand.

Only first in six dimensions can we form simple rational invariants that lead to a sensible variational principle. The volume factor now has weight 3, so the possible scalars are weight -3, and we have the possibilities

$$RF^{mn}F_{mn},\ R^{mnab}F_{mn}F_{ab},\ R^{mnab}F_{ma}F_{nb} \tag{45}$$

In contrast to the situation in 4-d, all of these will lead to second order equations for the g, and all are irreducible—no arbitrary factors will appear in the variational principle. We pick the first one. The others are unsuitable for reasons outlined in the Appendix. The unique variational principle is then

$$\int RW\sqrt{}d^6\omega = \int RW\,d\Omega = 0, \quad W = F^{mn}F_{mn} \tag{46}$$

We may immediately note the following points:

- Whatever the field equations are, they will be essentially unified in A and g because the integrand is irreducible.
- One might imagine a situation in which the electromagnetic scalar is more or less constant in a small area of space-time—variation will then reduce to the usual Hilbert action.
- Likewise, in a small region in which R is roughly constant, we are left

with the Lorentz action of electrodynamics, which, as we have seen, will generate the full inhomogeneous Maxwell–Lorentz theory in space-time.

4. THE CALIBRATION INVARIANT FIELD EQUATIONS

The field equations for A are easily obtained by cookbook procedure, using the formula above for the Ricci scalar, so the expanded Lagrange density is

$$\begin{aligned}\mathcal{L} = (R^* + 5g^{mn}\partial_m A_n + 5A_n(g^{mn}\partial_m \log\sqrt{} + \partial_m g^{mn} + g^{mn}A_n))\\ \cdot g^{ma}g^{nb}\partial_{[m}A_{n]}\partial_{[a}A_{b]}\sqrt{}\end{aligned} \quad (47)$$

The Euler–Lagrange equations with respect to A are

$$5W(g^{mn}\partial_m \log\sqrt{} + \partial_m g^{mn} + 2g^{mn}A_n) - \partial_n(5W\sqrt{}g^{mn} + 4R\sqrt{}F^{mn}) = 0 \quad (48)$$

or

$$\partial_m(\sqrt{}RF^{mn}) = -\frac{5}{4}\sqrt{}g^{mn}(\nabla_m - 2A_m)W = \frac{5}{4}\sqrt{}C^m \quad (49)$$

We see here the appearance of a *purely geometrical* charge current on the right that is a proper absolute invariant. The equations have the proper form of Maxwell's equations in curved space-time, but the appearance of R under the differentiation on the left shows immediately that the g and A fields are essentially coupled. Despite their simple form, these equations are in fact rather complex because of the implicit dependence of R on A.

It is remarkable that without ever introducing electrons, we have recovered the essential elements of electrodynamics, justifying Einstein's famous statement: "Das Elektron ist ein Fremder in die Elektrodynamik!"—the electron is a stranger in electrodynamics!

The conservation of "geometric charge" has an interesting expression in terms of the conformal covariant derivative. We have

$$\frac{5}{4}\partial_n(\sqrt{}C^n) = \partial_m\partial_n(\sqrt{}RF^{mn}) = 0 \quad (50)$$

which may be invariantly written as

$$D_m D^m W = 0 \quad (51)$$

This amounts to a "conformal wave equation" for W. Note that the equation is exact—this is not a flat space approximation. In terms of the ordinary covariant derivative it is

$$\begin{aligned}(\nabla_m - 3A_m)g^{mn}(\nabla_n - 2A_n)W = 0\\ g^{mn}(\nabla_m - 2A_m)(\nabla_n - 2A_n)W = 0\end{aligned} \quad (52)$$

To get the gravitational equations, we employ the Palatini procedure, which holds as well in Weyl space as it does in Riemann space, because it is not based on the particular form of the connection, rather the simple observation that the difference of two connections is a tensor. We write

$$\begin{aligned}\delta R_{mn} &= \left(\partial_n \delta\Gamma^a_{ma} - \Gamma^a_{mn}\delta\Gamma^b_{ab}\right) - \left(\partial_a \delta\Gamma^a_{mn} - \Gamma^a_{mb}\delta\Gamma^b_{an} - \Gamma^a_{bn}\delta\Gamma^b_{am} + \Gamma^a_{ab}\delta\Gamma^b_{mn}\right) \\ &= \left(\delta\Gamma^a_{mn}\right)_{;n} - \left(\delta\Gamma^a_{mn}\right)_{;a} \end{aligned} \tag{53}$$

We do the variation in several steps. First write

$$\delta(RW\surd) = W\delta(R\surd) + R\delta(W\surd) - RW\delta\surd \tag{54}$$

We have

$$\begin{aligned}\delta(R\surd) &= R_{rs}\delta(g^{rs}\surd) + g^{rs}\surd\delta R_{rs} \\ &= R_{rs}\delta(g^{rs}\surd) + g^{rs}\surd\left(\delta\Gamma^a_{ra}\right)_{;s} - g^{rs}\surd\left(\delta\Gamma^a_{rs}\right)_{;a} \\ &= R_{rs}\delta(g^{rs}\surd) + \surd\left(g^{rs}\delta\Gamma^a_{ra}\right)_{;s} - \surd\left(g^{rs}\delta\Gamma^a_{rs}\right)_{;a} \\ &\quad - \surd\left(g^{rs}_{;s}\delta\Gamma^a_{ra}\right) + \surd\left(g^{rs}_{;a}\delta\Gamma^a_{rs}\right) \\ &= R_{rs}\delta(g^{rs}\surd) + \surd\left(g^{rs}\delta\Gamma^a_{ra}\right)_{;s} - \surd\left(g^{rs}\delta\Gamma^a_{rs}\right)_{;a} \\ &\quad - \surd\left(g^{rs}A_s\delta\Gamma^a_{ra}\right)_{\surd}\left(g^{rs}A_a\delta\Gamma^a_{rs}\right) \\ &= R_{rs}\delta(g^{rs}\surd) + \partial_s\left(\surd g^{rs}\delta\Gamma^a_{ra}\right) - \partial_a\left(\surd g^{rs}\delta\Gamma^a_{rs}\right) \\ &\quad + 2\surd g^{rs}A_s\delta\Gamma^a_{ra} - 2\surd g^{rs}A_a\delta\Gamma^a_{rs} \end{aligned} \tag{55}$$

Above, we used the following conversion of a covariant divergence of a vector to an ordinary divergence in a 6-d Weyl space:

$$\begin{aligned} V^\mu_{;\mu} &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu}V^\nu = \partial_\mu V^\mu + \left(\frac{1}{\surd}\partial_\mu\surd + 3A_\mu\right)V^\mu \\ \surd V^\mu_{;\mu} &= \partial_\mu(\surd V^\mu) + \surd 3A_\mu V^\mu \end{aligned} \tag{56}$$

Proceeding, we have, using the above definition of the geometric current, the following removal of a divergence by Stoke's theorem and parts integration (indicated by the wavy equality sign):

$$\begin{aligned} W\delta(R\surd) &= WR_{rs}\delta(g^{rs}\surd) + W\partial_s\left(\surd g^{rs}\delta\Gamma^a_{ra}\right) - W\partial_a\left(\surd g^{rs}\delta\Gamma^a_{rs}\right) \\ &\quad + \surd g^{rs}\delta\Gamma^a_{ra}2WA_s - \surd g^{rs}\delta\Gamma^a_{rs}2WA_a \\ &\approx WR_{rs}\delta(g^{rs}\surd) + \surd g^{rs}\delta\Gamma^a_{ra}C_s - \surd g^{rs}\delta\Gamma^a_{rs}C_a \\ &= \left(WR_{rs} - \Gamma^a_{ra}C_s + \Gamma^a_{rs}C_a\right)\delta(g^{rs}\surd) + \delta\left(\surd g^{rs}\Gamma^a_{ra}\right)C_s - \left(\delta\surd g^{rs}\Gamma^a_{rs}\right)C_a \end{aligned} \tag{57}$$

We must now express the variation of the connection in terms of the variation of the g field. We use the following identities:

$$\begin{aligned}\partial_r(\surd g^{ab}) &= \left(-g^{as}\Gamma^b_{sr} - g^{bs}\Gamma^a_{sr} + g^{ab}\Gamma^s_{sr} - 2g^{ab}A_r\right)\surd\\ \partial_b(\surd g^{ab}) &= \left(-g^{rs}\Gamma^a_{sr} - 2A^a\right)\surd\\ \delta\surd &= \frac{1}{4}g_{mn}\delta(\surd g^{mn})\end{aligned}\tag{58}$$

Returning to the development, we have, noting that A is not varied and again removing a divergence:

$$\begin{aligned}\delta\left(\surd g^{rs}\Gamma^a_{ra}\right)C_s &= \delta(g^{rs}\partial_r\surd + 3\surd g^{rs}A_r)C_s\\ &= \delta(g^{rs}\partial_r\surd)C_s + 3A_r\delta(\surd g^{rs})C_s\\ &= (\delta_r\surd)\delta g^{rs}C_s + g^{rs}C_s\partial_r(\delta\surd) + 3A_rC_s\delta(\surd g^{rs})\\ &= (\partial_r\surd)\delta g^{rs}C_s + g^{rs}C_s\partial_r\left(\frac{1}{4}g_{mn}\delta(\surd G^{mn})\right) + 3A_rC_s\delta(\surd g^{rs})\\ &\approx (\partial_r\surd)\delta g^{rs}C_s - \partial_r(g^{rs}C_s)\frac{1}{4}g_{mn}\delta(\surd g^{mn}) + 3A_rC_s\delta(\surd g^{rs})\end{aligned}\tag{59}$$

We need to re-express the term in the direct variation of g:

$$\begin{aligned}\surd\delta g^{rs} &= \delta(\surd g^{rs}) - g^{rs}\frac{1}{4}g_{mn}\delta(\surd g^{mn}) = \left(\delta^r_m\delta^s_n - \frac{1}{4}g^{rs}g_{mn}\right)\delta(\surd g^{mn})\\ \delta g^{rs} &= \frac{1}{\surd}\left(\delta^r_m\delta^s_n - \frac{1}{4}g^{rs}g_{mn}\right)\delta(\surd g^{mn})\end{aligned}\tag{60}$$

and so, using the previously derived electromagnetic field equations,

$$\begin{aligned}\delta\left(\surd g^{rs}\Gamma^a_{ra}\right)C_s &= \left((\partial_m\log\surd)C_n\left(\delta^m_r\delta^n_s - \frac{1}{4}g^{mn}g_{rs}\right)\right.\\ &\qquad\left. - \partial_m(g^{mn}C_n)\frac{1}{4}g_{rs} + 3A_rC_s\right)\delta(\surd g^{rs})\\ &= \left((\partial_r\log\surd)C_s - \frac{1}{4}g_{rs}\frac{1}{\surd}\partial_m(\surd C^m) + 3A_rC_s\right)\delta(\surd g^{rs})\\ &= ((\partial_r\log\surd)C_s + 3A_rC_s)\delta(\surd g^{rs})\end{aligned}\tag{61}$$

Likewise we have after removal of a divergence:

$$\begin{aligned}\delta\left(\surd g^{rs}\Gamma^a_{rs}\right)C_a &= \delta(\partial_s(\surd g^{rs}) + 2\surd g^{rs}A_s)C_r\\ &= C_r\delta\partial_s(\surd g^{rs}) + 2A_sC_r\delta(\surd g^{rs})\\ &\approx (-\partial_sC_r + 2A_sC_r)\delta(\surd g^{rs})\end{aligned}\tag{62}$$

so at length we arrive at

$$\begin{aligned} W\delta(R\surd) &= (WR_{rs} - \Gamma^a_{ra}C_s + \Gamma^a_{rs}C_a + (\partial_r \log\surd)C_s - \partial_s C_r \\ &\quad + 3A_r C_s + 2A_s C_r)\delta(\surd g^{rs}) \\ &= \big(WR_{rs} + \big((\partial_r \log\surd) + 3A_r - \Gamma^a_{ra}\big)C_s - C_{r;s} + 2A_s C_r\big)\delta(\surd g^{rs}) \\ &= (WR_{rs} - C_{r;s} + 2A_s C_r)\delta(\surd g^{rs}) \end{aligned} \tag{63}$$

Next, we have

$$\begin{aligned} R\delta(W\surd) - RW(\delta\surd) &= R\delta(g^{mn}g^{rs}F_{mr}F_{ns}\surd) - RW(\delta\surd) \\ &= R(2g^{mn}F_{mr}F_{ns}\delta(\surd g^{rs}) - 2W\delta\surd) \\ &= -2R\left(F_{rn}F^n_s + \frac{1}{4}Wg_{rs}\right)\delta(\surd g^{rs}) = -2RT_{rs}\delta(\surd g^{rs}) \end{aligned} \tag{64}$$

where T is the usual electromagnetic energy-momentum tensor. Collecting everything we now have

$$\delta(RW\surd) = (WR_{rs} - 2RT_{rs} - C_{r;s} + 2A_s C_r)\delta(\surd g^{rs}) \tag{65}$$

Recall the Ricci tensor is not symmetric—thus, in this expression, we may add a symmetrizing term inside that is antisymmetric without altering the variation. We find

$$\begin{aligned} C_{r;s} - 2A_s C_r + WF_{rs} &= C_{r;s} - 2\left(\frac{C_s + \partial_s W}{W}\right)C_r \\ &\quad + W\partial_{[r}\left(\frac{C_{s]} + \partial_{s]}W}{2W}\right) \\ &= \frac{1}{2}C_{\{r,s\}} - C_{\{r}\left(\frac{C_{s\}} + \partial_{s\}}W}{2W}\right) \\ &= \frac{1}{2}(\nabla_{\{r} - 2A_{\{r})C_{s\}} \\ &= -\frac{1}{2}(\nabla_{\{r} - 2A_{\{r})(\nabla_{s\}} - 2A_{s\}})W \end{aligned} \tag{66}$$

and so finally

$$\delta(RW\surd) = \left(WR_{\underline{rs}} - 2RT_{rs} + \frac{1}{2}(\nabla_{\{r} - 2A_{\{r}\right)(\nabla_{s\{} - 2A_{s\})}W)\delta(\surd g^{rs}) \tag{67}$$

expressing the variation of the integrand in terms of the arbitrary variation of the metric. The underbars on the indices of the Ricci tensor indicate the symmetric part. Thus the combined electromagnetic and gravitational equations are

$$\begin{gathered} R_{\underline{mn}} = \left(\frac{2R}{W}\right) T_{mn} - \left(\frac{1}{2W}\right)(D_m D_n + D_n D_m)W \\ \frac{1}{\surd}\partial_n(\surd R F^{mn}) = \frac{5}{4} D^m W \end{gathered} \tag{68}$$

where D is the conformal covariant derivative introduced above. Because T has conformal weight -1, the equations are calibration invariant by inspection.

Under what conditions do these equations take on the form of Einstein's equations? We write

$$\begin{aligned} R_{\underline{mn}} - \frac{1}{2} g^{mn}(R - \Lambda) = \left(\frac{2R}{W}\right) \Bigg(& T_{mn} - \frac{1}{4} g_{mn} \frac{W}{R}(R - \Lambda) \\ & - \left(\frac{1}{4R}\right)(D_m D_n + D_n D_m)W \Bigg) \end{aligned} \tag{69}$$

The last two terms on the right must cancel:

$$(D_m D_n + D_n D_m - g_{mn}(R - \Lambda))W = 0 \tag{70}$$

Contracting with the metric and remembering the conformal wave equation for W we find

$$(R - \Lambda)W = 0 \tag{71}$$

Thus, we obtain general relativity only in the limit

$$\Lambda = 0,\ R \to 0,\ W \to 0,\ \frac{R}{W} \to -4\pi G \tag{72}$$

It thus appears that the indeterminate aspect of the Einstein equations represented by the arbitrary cosmological constant, is an artifact of the decoupling of gravity and electromagnetism. If R and W differ from zero by a factor of first order, their product is second order and may be ignored. Thus from the current perspective, the Einstein–Maxwell equations are to be regarded as a first-order approximation to the full calibration-invariant system.

One striking feature of these equations that distinguishes them from Einstein's equations is the absent gravitational constant—in fact the ratio of scalars in front of the energy tensor plays that role. This explains the odd role of G in general relativity and its scaling behavior (see Weinberg, 1972). The ratio has conformal weight l and so G has a natural dimensionfulness that prevents it from being a proper coupling constant—so this theory explains why ordinary general relativity, even in the linear approximation and the quantum theory built on it, cannot be regularized.

In other papers we take up this issue of finding simple solutions to these equations, as well as the possibility of extending the idea of conformal covariance to spinor fields. We also go into specific detail regarding solutions to the reformulation of electrodynamics presented above and its interpretation.

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APPENDIX

The Possible Scalars for a Varitional Principle

The following scalars

$$R^{mnab} F_{mn} F_{ab},\ R^{mnab} F_{ma} F_{nb} \tag{73}$$

are rejected for algebraic reasions. Recalling

$$R_{mnab} + R_{nmap} = g_{mn} F_{ab} \tag{74}$$

which is the signature of the non-Riemannian character of Weyl space, we see that the first scalar projects away this symmetric part, while in the second case

$$R^{mnab} F_{ma} F_{nb} = R^{abmn} F_{ma} F_{nb} \tag{75}$$

again projecting out the non-Riemannina asymmetry of the curvature tensor. Therefore the particular choice for the integrand in the variational principle is essentially unique. Moreover this form leads to the following simple comparison with the usual formalism.

We assume R and W differ but little from some constant values,

$$R = R_0 + \frac{\epsilon}{W_0} R_1, \quad W = W_0 + \frac{\epsilon}{R_0} W_1 \tag{76}$$

where ϵ is assumed small compared to the first-order values. Then to first order

$$RW\sqrt{} = R_0 W_0 \sqrt{} + \epsilon (R_1 + W_1) \sqrt{} \tag{77}$$

and so the variation with fixed boundary is

$$\begin{aligned} \delta \sqrt{} RW \, d\Omega &= \delta \sqrt{} R_0 W_0 \, d\Omega + \delta \sqrt{} \epsilon (R_1 + W_1) \, d\Omega \\ &= R_0 W_0 \, \delta\Omega + \delta \sqrt{} \epsilon (R_1 + W_1) \, d\Omega \\ &= 0 + \delta \sqrt{} \epsilon (R_1 + W_1) \, d\Omega \end{aligned} \tag{78}$$

so we recover the usual combined Maxwell and Hilbert actions.

REFERENCES

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, p. 7.1, p. 10.8.
Weyl, H. (1917). *Annalen der Physik* **54**, 121–125, 133.
Weyl, H. (1918). *Sitzungsb. Akad. der Wiss. Berlin*, 465–480.
Weyl, H. (1952). *Space-Time-Matter*, Dover, reprint, sec. 35, New York.